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Decay of a bound state under a time-periodic perturbation: a toy case

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Abstract

We study the time evolution of a three-dimensional quantum particle, initially in a bound state, under the action of a time-periodic zero range interaction with 'strength' $\alpha(t)$. Under very weak generic conditions on the Fourier coefficients of $\alpha(t)$, we prove complete ionization as $t \to \infty$. We prove also that, under the same conditions, all the states of the system are scattering states.

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1. Introduction

In this paper we shall study the asymptotically complete ionization of a system given by a quantum particle interacting with a time-dependent singular potential in three dimensions. The Hamiltonian of the system is formally

$$H(t) = H_0 + H_I(t)$$

where H_0 is a zero range perturbation at the origin of the Laplacian, i.e.

$$\mathcal{D}(H_0) = \left\{\Psi \in L^2(\mathbb{R}^3) \mid \Psi(\boldsymbol{x}) = \Phi(\boldsymbol{x}) + \gamma^{-1}\Phi(0)\mathcal{G}_0(\boldsymbol{x}),\ \Phi \in H^2_{\mathrm{loc}}(\mathbb{R}^3),\ \Delta\Phi \in L^2(\mathbb{R}^3)\right\}$$
$$H_0\Psi = -\Delta\Phi \tag{1.1}$$

where $\mathcal{G}_0(\boldsymbol{x})$ is the Green function of the Laplacian,

$$\mathcal{G}_0(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}$$

and $H_I(t)$ is heuristically given by $\alpha(t)\delta(\boldsymbol{x} - \boldsymbol{r})$ where $\boldsymbol{r} \in \mathbb{R}^3\backslash\{0\}$ and $\alpha(t)$ is a periodic function with period T.

These kind of models have been widely studied (see e.g. [2–8]) as toy models of more complicated physical problems, such as strong laser ionization of Rydberg atoms

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or dissociation of molecules. Indeed, time-dependent point interactions are an interesting example of time-dependent perturbations that are not small in any sense with respect to the unperturbed Hamiltonian, so that time-dependent perturbation theory cannot be applied. On the other hand, since such models are solvable, namely all the spectral and scattering data can be explicitly calculated, the problem of asymptotically complete ionization can be studied in a non-perturbative way. Indeed one can explicitly prove that, starting at time $t = 0$ from a bound state φ of the system, the survival probability

$$|\theta(t)|^2 = |(\varphi, U(t,0)\varphi)|^2$$

has a power law decay to zero as $t \to \infty$ (see [3, 4] and references therein).

Essentially using Laplace transform techniques (for a review of the methods used, we shall refer to [3]), we shall prove that the system shows asymptotically complete ionization under suitable generic conditions on the Fourier coefficients of $\alpha(t)$ and that the survival probability has a power law decay for large time.

We stress the non-perturbative nature of the result. Indeed the complete ionization does not depend on the size of $\alpha(t)$ and it holds even if $\alpha(t)$ is very big (so that the time-dependent perturbation is small—in the sense of quadratic forms—with respect to the unperturbed Hamiltonian) or very small (so that the perturbation is large) or fast oscillating. Moreover the asymptotic behaviour is independent of the period T. In particular there is asymptotically complete ionization, even if the period is very large, as for time-adiabatic perturbations.

In section 2 we shall introduce the model, the equations for the coefficients $q^{(j)}(t)$ and their Laplace transforms, which will be the main objects under investigation. Applying the analytic Fredholm theorem to such equations, in sections 3, 4 and 5 we shall identify the singularities of their solutions on the closed right half plane; in section 6 we shall derive the main results about ionization.

2. The model

The model we are going to study describes a quantum particle subjected to a time-dependent zero range interaction. In the absence of the time-periodic perturbation, the Hamiltonian describes a zero range interaction placed at the origin. The strength of the interaction, i.e. the parameter γ in (1.1), is assumed to be $-1/4\pi$, in order to simplify the calculations, but the results do not depend on this choice. This system has a bound state of energy -1 and normalized eigenfunction

$$\Psi_0(\boldsymbol{x}) = \frac{\mathrm{e}^{-|\boldsymbol{x}|}}{\sqrt{4\pi}|\boldsymbol{x}|}. \tag{2.1}$$

The remaining part of the spectrum is absolutely continuous and coincides with $\mathbb{R}^+$.

The time-dependent perturbation is a zero range interaction placed at a point $\boldsymbol{r} \neq 0$ and with time-periodic strength $\alpha(t)$ with period T.

The entire system is then described (see [1]) by the time-dependent self-adjoint Hamiltonian $H(t)$, with domain

$$\mathcal{D}(H(t)) = \big\{\Psi_t(\boldsymbol{x}) = \Phi(\boldsymbol{x};t) + q^{(1)}(t)\mathcal{G}_0(\boldsymbol{x}) + q^{(2)}(t)\mathcal{G}_0(\boldsymbol{x}-\boldsymbol{r}),$$
$$\Phi \in H^2_{\mathrm{loc}}(\mathbb{R}^3), \Delta\Phi \in L^2(\mathbb{R}^3)\big\} \tag{2.2}$$

$$\Phi(0;t) + q^{(2)}(t)\mathcal{G}_0(\boldsymbol{r}) = -\frac{q^{(1)}(t)}{4\pi} \qquad \Phi(\boldsymbol{r};t) + q^{(1)}(t)\mathcal{G}_0(\boldsymbol{r}) = \alpha(t)q^{(2)}(t). \tag{2.3}$$

Moreover for any $\Psi_t(\boldsymbol{x}) \in \mathcal{D}(H(t))$ one has

$$H(t)\Psi_t(\boldsymbol{x}) = -\Delta\Phi(\boldsymbol{x};t). \tag{2.4}$$

We want to stress that definitions (2.2) and (2.3) imply that the functions in $\mathcal{D}(H(t))$ have the following behaviour near the centres of the interactions

$$\lim_{\boldsymbol{x}\to 0}\{4\pi|\boldsymbol{x}|\Psi_t(\boldsymbol{x})\} = q^{(1)}(t) \qquad \lim_{\boldsymbol{x}\to \boldsymbol{r}}\{4\pi|\boldsymbol{x}-\boldsymbol{r}|\Psi_t(\boldsymbol{x})\} = q^{(2)}(t).$$

It is well known (see [9–11, 13, 14]) that the solution of the time-dependent Schrödinger equation

$$\mathrm{i}\frac{\partial \Psi_t}{\partial t} = H(t)\Psi_t \tag{2.5}$$

associated with operator (2.4) is given by

$$\Psi_t(\boldsymbol{x}) = U_0(t-s)\Psi_s(\boldsymbol{x}) + \mathrm{i}\int_s^t \mathrm{d}\tau [q^{(1)}(\tau)U_0(t-\tau;\boldsymbol{x}) + q^{(2)}(\tau)U_0(t-\tau;\boldsymbol{x}-\boldsymbol{r})] \tag{2.6}$$

where $U_0(t) = \exp(\mathrm{i}\Delta t)$, $U_0(t;\boldsymbol{x})$ is the kernel associated with the free propagator and the functions $q^{(j)}(t)$ satisfy a system of Volterra integral equations for $t \geqslant s$,

$$\begin{aligned} q^{(1)}(t) + \frac{\sqrt{-2\mathrm{i}}}{\pi}\int_s^t \mathrm{d}\tau q^{(2)}(\tau)\int_\tau^t \mathrm{d}\sigma \frac{U_0(\sigma-\tau;\boldsymbol{r})}{\sqrt{t-\sigma}} \\ - \frac{1}{\sqrt{-\pi\mathrm{i}}}\int_s^t \mathrm{d}\tau \frac{q^{(1)}(\tau)}{\sqrt{t-\tau}} = 4\sqrt{\pi\mathrm{i}}\int_s^t \mathrm{d}\tau \frac{(U_0(\tau)\Psi_s)(0)}{\sqrt{t-\tau}} \end{aligned} \tag{2.7}$$

$$\begin{aligned} q^{(2)}(t) + \frac{\sqrt{-2\mathrm{i}}}{\pi}\int_s^t \mathrm{d}\tau q^{(1)}(\tau)\int_\tau^t \mathrm{d}\sigma \frac{U_0(\sigma-\tau;\boldsymbol{r})}{\sqrt{t-\sigma}} \\ + 4\sqrt{\pi\mathrm{i}}\int_s^t \mathrm{d}\tau \frac{\alpha(\tau)q^{(2)}(\tau)}{\sqrt{t-\tau}} = 4\sqrt{\pi\mathrm{i}}\int_s^t \mathrm{d}\tau \frac{(U_0(\tau)\Psi_s)(\boldsymbol{r})}{\sqrt{t-\tau}}. \end{aligned} \tag{2.8}$$

We are interested in studying asymptotic complete ionization of system defined by (2.4) and (2.5), starting by the normalized bound state (2.1) at time $t = 0$. Moreover we shall require that $\alpha(t)$ be a real continuous periodic function with period T, so that it can be decomposed in a Fourier series, for each $t \in \mathbb{R}^+$, and the series converges uniformly on every compact subset of the real line. More precisely, in terms of Fourier coefficients of $\alpha(t)$, we assume

$$\begin{aligned} &(1) \quad \alpha(t) = \sum_{n\in\mathbb{Z}} \alpha_n \,\mathrm{e}^{-\mathrm{i}n\omega t}, \qquad \{\alpha_n\} \in \ell_1(\mathbb{Z}) \\ &(2) \quad \alpha_n = \alpha^*_{-n}. \end{aligned} \tag{2.9}$$

We now introduce a *generic* condition on $\alpha(t)$ that will be used later on. Let $\mathcal{T}$ be the right shift operator on $\ell_2(\mathbb{N})$, i.e.

$$(\mathcal{T}\alpha)_n \equiv \alpha_{n+1} \tag{2.10}$$

we say that $\alpha = \{\alpha_n\} \in \ell_2(\mathbb{Z})$ is *generic* with respect to $\mathcal{T}$, if $\tilde{\alpha} \equiv \{\alpha_n\}_{n>0} \in \ell_2(\mathbb{N})$ satisfies the following condition:

$$e_1 = (1, 0, 0, \ldots) \in \overline{\bigvee_{n=0}^{\infty} \mathcal{T}^n \tilde{\alpha}}. \tag{2.11}$$

For a detailed discussion of genericity condition see [4]. Note that

$$\alpha_0 \equiv \frac{1}{T}\int_0^T \alpha(t)\,\mathrm{d}t \tag{2.12}$$

does not enter in the condition.

By simple estimates on the sup norm of $r_j(t) \equiv q^{(j)}(t)\,\mathrm{e}^{-bt}$, it is easy to prove that $q^{(j)}(t)$ has at most an exponential behaviour as $t \to \infty$, i.e. asymptotically $|q^{(j)}(t)| \leqslant A_j\,\mathrm{e}^{b_j t}$.

Therefore the Laplace transform of $q^{(j)}(t)$, denoted by

$$\tilde{q}^{(j)}(p) \equiv \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-pt} q^{(j)}(t)$$

exists and is analytic at least for $\mathrm{Re}(p) > b_0$. Hence, applying the Laplace transform to equations (2.7) and (2.8), one has

$$\tilde{q}^{(1)}(p) = -\frac{1}{(2\pi)^{\frac{3}{2}} r} \frac{\mathrm{e}^{-r\sqrt{-\mathrm{i}p}}}{1-\sqrt{-\mathrm{i}p}} \tilde{q}^{(2)}(p) + F_1(p) \tag{2.13}$$

$$\tilde{q}^{(2)}(p) = -\frac{4\pi}{\sqrt{-\mathrm{i}p}} \sum_{k\in\mathbb{Z}} \alpha_k \tilde{q}^{(2)}(p+\mathrm{i}\omega k) + \frac{\mathrm{e}^{-\sqrt{-\mathrm{i}p}r}}{2\pi r\sqrt{-2\pi\mathrm{i}p}} \tilde{q}^{(1)}(p) + F_2(p) \tag{2.14}$$

where the explicit expression of $F_i(p)$ for the initial datum (2.1) is given by

$$F_1(p) \equiv -\frac{2\mathrm{i}\sqrt{2\pi}}{1+\mathrm{i}p}$$

$$F_2(p) \equiv -\frac{2\mathrm{i}\sqrt{2\pi}}{\sqrt{-\mathrm{i}p}} \frac{\mathrm{e}^{-\sqrt{-\mathrm{i}p}r} - \mathrm{e}^{-r}}{r(1+\mathrm{i}p)}.$$

Let us start by considering the system of equations (2.13) and (2.14), for the specific initial datum (2.1): analyticity at least for $\mathrm{Re}(p) > b_0$ suggests to choose the branch cut of the square root along the negative real line: if $p = \varrho\,\mathrm{e}^{\mathrm{i}\vartheta}$,

$$\sqrt{p} = \sqrt{\varrho}\,\mathrm{e}^{\mathrm{i}\vartheta/2} \tag{2.15}$$

with $-\pi < \vartheta \leqslant \pi$.

Before dealing with the behaviour of the solution, let us simplify the problem: setting $q_n^{(j)}(p) \equiv \tilde{q}^{(j)}(p+\mathrm{i}\omega n)$ we obtain a sequence of functions on the strip $\mathcal{I} = \{p \in \mathbb{C}, 0 \leqslant \mathrm{Im}(p) < \omega\}$ and setting $q_j(p) \equiv \{q_n^{(j)}(p)\}_{n\in\mathbb{Z}}$, equations (2.13) and (2.14) can be rewritten as

$$q_1(p) = \mathcal{M}_1 q_2(p) + G_1(p) \tag{2.16}$$

$$q_2(p) = \mathcal{L} q_2(p) + \mathcal{M}_2 q_1(p) + G_2(p) \tag{2.17}$$

where

$$(\mathcal{M}_1 q)_n(p) \equiv -\frac{1}{(2\pi)^{\frac{3}{2}} r} \frac{\mathrm{e}^{-r\sqrt{\omega n - \mathrm{i}p}}}{1-\sqrt{\omega n - \mathrm{i}p}} q_n(p) \tag{2.18}$$

$$(\mathcal{M}_2 q)_n(p) \equiv \frac{1}{(2\pi)^{\frac{3}{2}} r} \frac{\mathrm{e}^{-r\sqrt{\omega n - \mathrm{i}p}}}{4\pi\alpha_0 + \sqrt{\omega n - \mathrm{i}p}} q_n(p) \tag{2.19}$$

$$(\mathcal{L} q)_n(p) \equiv -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - \mathrm{i}p}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k q_{n+k}(p) \tag{2.20}$$

and $G_j(p) = \{g_n^{(j)}(p)\}_{n\in\mathbb{Z}}$ with

$$g_n^{(1)}(p) \equiv \frac{2\mathrm{i}\sqrt{2\pi}}{1-\omega n + \mathrm{i}p} \tag{2.21}$$

$$g_n^{(2)}(p) \equiv -\frac{2\mathrm{i}\sqrt{2\pi}}{r} \frac{\mathrm{e}^{-r\sqrt{\omega n - \mathrm{i}p}} - \mathrm{e}^{-r}}{(4\pi\alpha_0 + \sqrt{\omega n - \mathrm{i}p})(1-\omega n + \mathrm{i}p)}. \tag{2.22}$$

3. Analyticity on the (open) right half plane

Let us extend equations (2.16) and (2.17) on the whole open right half plane: we are going to prove that the solution exists and is analytic for $\mathrm{Re}(p) > 0$. Let us start with some preliminary results:

Proposition 3.1. *For $p \in \mathcal{I}, \mathrm{Re}(p) > 0, \mathcal{M}_j(p)$ are analytic operator-valued functions and $\mathcal{M}_j(p)$ are compact operators on $\ell_2(\mathbb{Z})$.*

Proof. Let us consider only $\mathcal{M}_1$, since the argument also applies to $\mathcal{M}_2$. The analyticity of the operator is a straightforward consequence of the explicit expression (2.18). Moreover the operator $\mathcal{M}_1(p)$ is a multiplication operator in $\ell_2(\mathbb{Z})$ and it is bounded and compact since

$$\left\{ \frac{1}{(2\pi)^{\frac{3}{2}} r} \frac{\mathrm{e}^{-r\sqrt{\omega n - \mathrm{i}p}}}{1 - \sqrt{\omega n - \mathrm{i}p}} \right\} \in \ell_2(\mathbb{Z})$$

on the open right half plane: indeed choice (2.15) for the branch cut of the square root implies $\mathrm{Re}(\sqrt{\omega n - \mathrm{i}p}) > 0$, if $\mathrm{Re}(p) > 0$. □

Proposition 3.2. *For $p \in \mathcal{I}, \mathrm{Re}(p) > 0, \mathcal{L}(p)$ is an analytic operator-valued function and $\mathcal{L}(p)$ is a compact operator on $\ell_2(\mathbb{Z})$.*

Proof. Analyticity for $\mathrm{Re}(p) > 0$ easily follows from the explicit expression of the operator. Moreover $\mathcal{L}(p)$ can be written as

$$\mathcal{L}(p) = A(p) \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k \mathcal{T}^{n+k}$$

where $A(p)$ is the operator

$$(Aq)_n(p) \equiv A_n(p) q_n(p) = -\frac{4\pi q_n(p)}{4\pi\alpha_0 + \sqrt{\omega n - \mathrm{i}p}}$$

and $\mathcal{T}$ is the right shift operator on $\ell_2(\mathbb{Z})$. Since $\|\mathcal{T}\| = 1$, the series converges strongly to a bounded operator. Moreover $A(p)$ is a compact operator for $\mathrm{Re}(p) > 0$: $A(p)$ is the norm limit of a sequence of finite rank operators, because $\lim_{n\to\infty} A_n(p) = 0$. Hence the result follows e.g. from theorems VI.12 and VI.13 of [12]. □

Lemma 3.1. *For each $r, \omega \in \mathbb{R}^+$ and for* $\mathrm{Re}(p) > 0$

$$\mathrm{Im}\left[\sqrt{\omega n - \mathrm{i}p} + \frac{1}{(2\pi)^3 r^2} \frac{\mathrm{e}^{-2r\sqrt{\omega n - \mathrm{i}p}}}{1 - \sqrt{\omega n - \mathrm{i}p}} \right] < 0$$

$\forall n \in \mathbb{Z}$.

Proof. First of all we want to stress that choice (2.15) for the branch cut implies that $\mathrm{Re}(\sqrt{\omega n - \mathrm{i}p}) > 0$ and $\mathrm{Im}(\sqrt{\omega n - \mathrm{i}p}) < 0$, if $\mathrm{Re}(p) > 0$. Calling $x \equiv \mathrm{Re}(\sqrt{\omega n - \mathrm{i}p})$, $y \equiv \mathrm{Im}(\sqrt{\omega n - \mathrm{i}p})$ and

$$f_r(x, y) \equiv \mathrm{Im}\left[x + \mathrm{i}y + \frac{1}{(2\pi)^3 r^2} \frac{\mathrm{e}^{-2r(x+\mathrm{i}y)}}{1 - x - \mathrm{i}y} \right]$$

one has

$$\left| \frac{1}{(2\pi)^3 r^2} \frac{\mathrm{e}^{-2r(x+\mathrm{i}y)}}{1 - x - \mathrm{i}y} \right| < \frac{1}{(2\pi)^3 r^2 |y|}$$

and then $f_r(x, y) \leqslant 0$, if $|y| \geqslant [(2\pi)^{3/2}r]^{-1}$. Moreover

$$f_r(x, y) = \frac{(2\pi)^3 r^2[(1-x)^2 + y^2]y + \mathrm{e}^{-2rx}[y\cos(2ry) - (1-x)\sin(2ry)]}{(2\pi)^3 r^2[(1-x)^2 + y^2]}$$

and the claim is true if $x \geqslant 1$, since $\sin(2ry) < 0$ and $\cos(2ry) > 0$, for $y > -[(2\pi)^{3/2}r]^{-1}$. Hence it is sufficient to prove that $f_r(x, y) < 0$ on the set

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x < 1, -[(2\pi)^{3/2}r]^{-1} < y < 0\}.$$

Now set

$$g_r(x, y) \equiv \frac{(2\pi)^3 r^2[(1-x)^2 + y^2] f_r(x, y)}{y}$$

and consider

$$\frac{\partial g_r}{\partial y} = 2(2\pi)^3 r^2 y - 2\,\mathrm{e}^{-2rx}\left[r\sin(2ry) + \frac{r(1-x)\cos(2ry)}{y} - \frac{(1-x)\sin(2ry)}{2y^2}\right].$$

Since, for $(x, y) \in \mathcal{R}, 2\,\mathrm{e}^{-2rx} r\sin(2r|y|) < 2(2\pi)^3 r^2|y|$ and $2ry\cos(2ry) \leqslant \sin(2ry)$ the partial derivative of g_r with respect to y is always negative in $\mathcal{R}$. Thus

$$g_r(x, y) \geqslant g_r(x, 0) > 0.$$

In conclusion $g_r(x, y) > 0$ and then $f_r(x, y) < 0, \forall (x, y) \in \mathcal{R}$. □

Proposition 3.3. *The solutions $\tilde{q}^{(j)}(p)$, $j = 1, 2$, of (2.13) and (2.14) are unique and analytic for* $\mathrm{Re}(p) > 0$.

Proof. Since $G_1(p) \in \ell_2(\mathbb{Z})$ is analytic on the right half plane and thanks to proposition 3.1, we can substitute (2.16) in (2.17) and consider only the second equation. Thus (2.17) now reads as

$$q_2(p) = [\mathcal{L} + \mathcal{M}_2\mathcal{M}_1]q_2(p) + \mathcal{M}_2 G_1(p) + G_2(p). \tag{3.1}$$

Then the key point will be the application of the analytic Fredholm theorem (theorem VI.14 of [12]) to the operator $\mathcal{L}'(p) \equiv \mathcal{L} + \mathcal{M}_2\mathcal{M}_1$, in order to prove that $(I - \mathcal{L}'(p))^{-1}$ exists for $\mathrm{Re}(p) > 0$.

So let us begin with the analysis of the homogeneous equation associated with (3.1),

$$q(p) = \mathcal{L}'(p)q(p)$$

and suppose that there exists a nonzero solution $Q(p) = \{Q_n(p)\}_{n\in\mathbb{Z}}$. Multiplying both sides of the equation by Q_n^* and summing over $n \in \mathbb{Z}$, we have

$$\sum_{n\in\mathbb{Z}}\left[\sqrt{\omega n - \mathrm{i}p} + \frac{1}{(2\pi)^3 r^2}\frac{\mathrm{e}^{-2r\sqrt{\omega n - \mathrm{i}p}}}{1 - \sqrt{\omega n - \mathrm{i}p}}\right]|Q_n|^2 = -4\pi\sum_{n,k\in\mathbb{Z}} Q_n^*\alpha_{k-n}Q_k$$

but, since the right-hand side is real, because of condition 2 in (2.9), it follows that

$$\mathrm{Im}\left[\sum_{n\in\mathbb{Z}}\left(\sqrt{\omega n - \mathrm{i}p} + \frac{1}{(2\pi)^3 r^2}\frac{\mathrm{e}^{-2r\sqrt{\omega n - \mathrm{i}p}}}{1 - \sqrt{\omega n - \mathrm{i}p}}\right)|Q_n|^2\right] = 0$$

and then, by lemma 3.1, $Q_n = 0, \forall n \in \mathbb{Z}$.

Since there is no nonzero solution of the homogeneous equation associated with (3.1) and $\mathcal{L}$ is compact on the whole open right half plane, analytic Fredholm theorem applies and the result then easily follows, because $\mathcal{M}_2 G_1(p) + G_2(p) \in \ell_2(\mathbb{Z})$ and, for each $n \in \mathbb{Z}$, $[\mathcal{M}_2 G_1(p) + G_2(p)]_n$ is analytic for $\mathrm{Re}(p) > 0$. □

4. Behaviour on the imaginary axis at $p \neq 0$

The equation for $q_2(p)$ can be written as

$$(4\pi\alpha_0 + c_n(p))q_n^{(2)}(p) = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k q_{n+k}^{(2)}(p) + f_n^{(2)}(p) \tag{4.1}$$

where

$$c_n(p) \equiv \sqrt{\omega n - \mathrm{i}p} + \frac{\mathrm{e}^{-2r\sqrt{\omega n - \mathrm{i}p}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega n - \mathrm{i}p})} \tag{4.2}$$

$$f_n^{(2)}(p) \equiv -\frac{2\mathrm{i}\sqrt{2\pi}}{r(1-\omega n + \mathrm{i}p)} \left[\frac{(2\pi)^{\frac{3}{2}} - 1}{(2\pi)^{\frac{3}{2}}} \, \mathrm{e}^{-r\sqrt{\omega n - \mathrm{i}p}} - \mathrm{e}^{-r} \right] \tag{4.3}$$

and it is clear that the solution may have a pole where

$$4\pi\alpha_0 + \sqrt{\omega n - \mathrm{i}p} + \frac{\mathrm{e}^{-2r\sqrt{\omega n - \mathrm{i}p}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega n - \mathrm{i}p})} = 0$$

and that the coefficients of the equation for $q_0^{(2)}$ fail to be analytic at $p = i$: for $p \in \mathcal{I}$, $\mathrm{Re}(p) = 0$, and $n \in \mathbb{Z}$, the unique solution of $1 - \sqrt{\omega n - \mathrm{i}p} = 0$ is $p = i$, $n = 0$.

In the following we shall see that in fact the solution is analytic on the imaginary axis except at most some singularity at $p = 0$. Let us start by considering the position of the eventual pole.

Lemma 4.1. *Assume that α_0 in (2.12) is non-negative. Then there exists a unique $n_0 \in \mathbb{N}$ and a unique $p_0 \in \mathcal{I}$, $\mathrm{Re}(p) = 0$, such that*

$$4\pi\alpha_0 + \sqrt{\omega n_0 - \mathrm{i}p_0} + \frac{\mathrm{e}^{-2r\sqrt{\omega n_0 - \mathrm{i}p_0}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega n_0 - \mathrm{i}p_0})} = 0.$$

Moreover $\forall n \in \mathbb{Z}$, $n < 0$ and $\forall p \in \mathcal{I}$, $\mathrm{Re}(p) = 0$,

$$\mathrm{Im}\left[4\pi\alpha_0 + \sqrt{\omega n - \mathrm{i}p} + \frac{\mathrm{e}^{-2r\sqrt{\omega n - \mathrm{i}p}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega n - \mathrm{i}p})} \right] > 0.$$

Proof. Let us first consider the second statement: on the strip $\mathcal{I}$ and for $n < 0$, $\sqrt{\omega n - \mathrm{i}p} \equiv \mathrm{i}\lambda$, with $\lambda \in \mathbb{R}$, $\lambda > 0$. Hence

$$\mathrm{Im}(c_n(\mathrm{i}\lambda)) = \frac{(2\pi)^3 r^2 (1+\lambda^2)\lambda + \lambda\cos(2r\lambda) - \sin(2r\lambda)}{(2\pi)^3 r^2 (1+\lambda^2)}$$

and following the proof of lemma 3.1, it can be easily proved that the expression above is positive $\forall \lambda \in \mathbb{R}^+$. On the other hand, if $n \geqslant 0$ and $p \in \mathcal{I}$, $\mathrm{Re}(p) = 0$, $\sqrt{\omega n - \mathrm{i}p} = \lambda$, with $\lambda > 0$, and, $\forall r, \omega \in \mathbb{R}^+$, the equation

$$(2\pi)^3 r^2 (4\pi\alpha_0 + \lambda)(\lambda - 1) = \mathrm{e}^{-2r\lambda}$$

has a unique solution for $\lambda \in \mathbb{R}^+$. Then, since there exists a unique $p_0 \in \mathcal{I}$, $\mathrm{Re}(p_0) = 0$, such that, for fixed $\lambda \in \mathbb{R}^+$, the equation $p_0 = \mathrm{i}(\lambda^2 - \omega n_0)$ is satisfied for some $n_0 \in \mathbb{N}$, the proof is complete. □

Lemma 4.2. *Assume that α_0 in (2.12) is non-negative and that $\{\alpha_n\}$ satisfies (2.9) and the genericity condition with respect to $\mathcal{T}$ (2.11). Then the solutions of (2.13) and (2.14) are unique and analytic on the imaginary axis for $p \neq 0, i, p_0$.*

Proof. Since for $p \in \mathcal{I}, \mathrm{Re}(p) = 0$ and $p \neq 0, i, p_0$, the coefficients of equations (2.16) and (2.17) are analytic (see lemma 4.1) and belong to $\ell_2(\mathbb{Z})$ and since the operators $\mathcal{L}, \mathcal{M}_1$ and $\mathcal{M}_2$ are still compact on the same region, it is sufficient to show that the homogeneous equation associated with (4.1) has no non-zero solution, in order to apply the analytic Fredholm theorem.

If Q_n is such a non-zero solution, following the proof of proposition 3.3, we immediately obtain the condition:

$$\sum_{n\in\mathbb{Z}} \left[\sqrt{\omega n - \mathrm{i}p} + \frac{1}{(2\pi)^3 r^2} \frac{\mathrm{e}^{-2r\sqrt{\omega n - \mathrm{i}p}}}{1 - \sqrt{\omega n - \mathrm{i}p}} \right] |Q_n|^2 \in \mathbb{R}$$

and then lemma 4.1 guarantees that $Q_n = 0, \forall n < 0$. Now let $n_1 \in \mathbb{N}$ be such that $Q_{n_1} \neq 0$. For $n < n_1$, one has $\sum_{k=n_1}^{\infty} \alpha_{k-n} Q_k = 0$ or, setting $k = n_1 - 1 + k'$, for $n \geqslant 0$,

$$\sum_{k'=1}^{\infty} \alpha_{k'+n} Q_{n_1-1+k'} = 0$$

and then, for each $n \geqslant 0$,

$$(Q', \mathcal{T}^n \alpha)_{\ell_2(\mathbb{N})} = 0$$

where $Q'_n = Q^*_{n_1-1+n}$ and $(\cdot, \cdot)$ stands for the standard scalar product on $\ell_2(\mathbb{N})$. Finally the genericity condition (2.11) implies that $Q'_1 = Q^*_{n_1} = 0$, which is a contradiction. Hence $Q_n = 0, \forall n \in \mathbb{Z}$. □

Proposition 4.1. *Assume that α_0 in (2.12) is non-negative and that $\{\alpha_n\}$ satisfies (2.9) and the genericity condition with respect to $\mathcal{T}$ (2.11). Then the solutions of (2.13) and (2.14) are unique and analytic on the imaginary axis except at most at $p = 0$.*

Proof. In the first part of the proof we are going to consider only equation (4.1) for $q_2(p)$ and we shall extend then the results to $q_1(p)$.

In order to prove analyticity of the solution we need to analyse the behaviour of the solution of (4.1) in a neighbourhood of $p = p_0$ (see lemma 4.1) and $p = i$ separately and show that it has no singularity, while, for $p \in \mathcal{I}, \mathrm{Re}(p) = 0$, and $p \neq i, p_0$, the result follows from lemma 4.2.

Let us look for a solution of (4.1) of the form (for simplicity we are going to omit the index 2)

$$q_n = u_n + v_n q_{n_0}$$

for $n \neq n_0$: q_n satisfies (4.1) if and only if $\{u_n\}, \{v_n\} \in \ell_2(\mathbb{Z}\backslash\{n_0\})$ are solutions of

$$c_n(p) u_n = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq n_0}} \alpha_{k-n} u_k + f_n^{(2)}(p) \tag{4.4}$$

$$c_n(p) v_n = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq n_0}} \alpha_{k-n} v_k - 4\pi \alpha_{n_0-n}. \tag{4.5}$$

Existence of non-zero solutions of the homogeneous equations associated with (4.4) and (4.5) can be excluded because of the genericity condition as in the proof of lemma 4.2 and then, since the coefficients of the equations above are analytic in a neighbourhood of p_0 and belong to $\ell_2(\mathbb{Z}\backslash\{n_0\})$, $\{u_n\}, \{v_n\} \in \ell_2(\mathbb{Z}\backslash\{n_0\})$ are analytic in the same neighbourhood.

Moreover q_{n_0} satisfies the equation

$$\left\{4\pi\alpha_0 + c_{n_0}(p) + 4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq n_0}} \alpha_{k-n_0} v_k \right\} q_{n_0} = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq n_0}} \alpha_{k-n_0} u_k + f_{n_0}^{(2)}(p).$$

It is then sufficient to show that

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq n_0}} \alpha_{k-n_0} v_k(p_0) \neq 0.$$

Let us suppose that the contrary is true: calling $V_n \equiv v_n(p_0)$, multiplying equation (4.5) at $p = p_0$ by V_n^* and summing over $n \in \mathbb{Z}, n \neq n_0$, one has

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq n_0}} \left\{ \sqrt{\omega n - \mathrm{i}p_0} + \frac{\mathrm{e}^{-2r\sqrt{\omega n - \mathrm{i}p_0}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega n - \mathrm{i}p_0})} \right\} |V_n|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq n_0}} V_n^* \alpha_{k-n} V_k.$$

Using condition (2.9) and the genericity condition (2.11), as in the proof of lemma 4.2, one obtains $V_n = 0, \forall n \in \mathbb{Z}\backslash\{n_0\}$, but this is impossible since V_n satisfies equation (4.5). This concludes the proof of analyticity of $q_2(p)$ in a neighbourhood of $p = p_0$. In the same way it is possible to conclude that $q_2(p)$ is also analytic at $p = i$.

It remains to study the behaviour of $q_1(p)$ and in particular to analyse $q_0^{(1)}(p)$ in a neighbourhood of $p = i$, where it may have a pole (see equation (2.16)): from (4.1) one has

$$\frac{\mathrm{e}^{-2r\sqrt{\omega n+1}}}{(2\pi)^3 r^2} q_n^{(2)}(i) = -\frac{2\mathrm{i}\sqrt{2\pi}}{r} \left[\frac{(2\pi)^{\frac{3}{2}} - 1}{(2\pi)^{\frac{3}{2}}} \mathrm{e}^{-r\sqrt{\omega n+1}} - \mathrm{e}^{-r} \right]$$

and then $q_0^{(1)}(i) = \mathrm{i}\sqrt{2\pi}$. □

Remark. Proposition 4.1 holds even if $\alpha_0 < 0$. The proof can be given in the same way but it is slightly more complicated, because $4\pi\alpha_0 + c_n(p)$ in lemma 4.1 could vanish in two points instead of one. Nevertheless the argument contained in proposition 4.1 can be applied once more, in order to exclude the presence of the corresponding singularity of the solution.

5. Behaviour at $p = 0$

We shall now study the behaviour of the solutions of (2.16) and (2.17) in a neighbourhood the origin. With choice (2.15) for the branch cut of the square root, it is clear that we must expect branch points of $\tilde{q}^{(j)}(p)$, solutions of (2.13) and (2.14), at $p = \mathrm{i}\omega n, n \in \mathbb{Z}$, which should imply a branch point at $p = 0$ for each $q_n^{(j)}$.

We are going to show that the solutions of (2.16) and (2.17) have a branch point singularity at the origin.

Proposition 5.1. *If $\{\alpha_n\}$ satisfies (2.9) and (2.11) (genericity condition), the solution of the system (2.13), (2.14) has the form $\tilde{q}^{(j)}(p) = c_j(p) + d_j(p)\sqrt{p}$, $j = 1, 2$, in an imaginary neighbourhood of $p = 0$, where the functions $c_j(p)$ and $d_j(p)$ are analytic at $p = 0$.*

Proof. The resonant case, namely if, for some $N \in \mathbb{N}$, $\omega = 1/N$, and the non-resonant one will be treated separately.

(1) Non-resonant case. Setting $q_n = u_n + v_n q_0$, $n \neq 0$ in (4.1), one obtains the following equations for $\{u_n\}, \{v_n\} \in \ell_2(\mathbb{Z}\backslash\{0\})$:

$$c_n(p) u_n = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_{k-n} u_k + g_n^{(2)}(p) \tag{5.1}$$

$$c_n(p) v_n = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_{k-n} v_k - 4\pi\alpha_{-n}. \tag{5.2}$$

If, for every $n \in \mathbb{Z}$, $c_n(0) \neq -4\pi\alpha_0$, using the genericity condition, it is easy to prove that $\{u_n\}, \{v_n\} \in \ell_2(\mathbb{Z}\backslash\{0\})$ are unique and analytic at $p = 0$. On the other hand if the condition above is not satisfied and there exists $N_1 \in \mathbb{Z}$ such that

$$4\pi\alpha_0 + \sqrt{\omega N_1} + \frac{\mathrm{e}^{-2r\sqrt{\omega N_1}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega N_1})} = 0$$

one can repeat the trick, setting for example $v_n = u'_n + v'_n v_{N_1}$ for $n \neq N_1$, and prove that in fact $\{u_n\}$ and $\{v_n\}$ are still analytic in a neighbourhood of $p = 0$. Thus it is sufficient to prove that q_0, which is solution of

$$\left\{ 4\pi\alpha_0 + c_0(p) + 4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k v_k \right\} q_0(p) = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k u_k + f_0^{(2)}(p)$$

has the required behaviour near $p = 0$. First, setting $v_n^0 = v_n(p = 0)$, we have to prove that

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k v_k^0 \neq -\alpha_0 - \frac{1}{4\pi(2\pi)^3 r^2}$$

but, assuming that the contrary is true and multiplying both sides of equation (5.2), with $n_0 = 0$, by v_n^{0*} and summing over $n \in \mathbb{Z}$, $n \neq 0$, one has

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \sqrt{\omega n} \left|v_n^0\right|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq 0}} v_n^{0*} \alpha_{k-n} v_k^0 + 4\pi\alpha_0 + \frac{1}{(2\pi)^3 r^2}$$

The right-hand side is still real so that, assuming that the genericity condition is satisfied by $\{\alpha_n\}$ and applying the argument contained in the proof of proposition 4.1, we immediately obtain $\left\{v_n^0\right\} = 0$, which is a contradiction, since $\left\{v_n^0\right\}$ solves (5.2). The result for $\tilde{q}^{(2)}$ follows then directly from the equation for q_0, since $\mathrm{e}^{-2r\sqrt{-\mathrm{i}p}}$ has a branch cut along the negative real line. The extension to $q^{(1)}$ is thus trivial.

(2) Resonant case. As before let us look for a solution of (4.1) of the form $q_n = u_n + v_n q_0$, $n \neq 0$, so that $\{u_n\}, \{v_n\} \in \ell_2(\mathbb{Z}\backslash\{0\})$ solve (5.1) and (5.2) with $\omega = 1/N$. Multiplying both sides of (5.1) and (5.2) for $n = N$ by $1 - n/N - \mathrm{i}p$, one sees that u_N and v_N have no pole singularity at $p = 0$. On the other hand, if there exists $N_1 \in \mathbb{Z}$ such that

$$4\pi\alpha_0 + \sqrt{\frac{N_1}{N}} + \frac{\mathrm{e}^{-2r\sqrt{N_1/N}}}{(2\pi)^3 r^2 (1 - \sqrt{N_1/N})} = 0$$

the solutions could have a pole at $p = 0$, for $n = N_1$ (the expression above guarantees that $N_1 \neq N$). Nevertheless, repeating the above procedure for $n = N_1$, it is easily seen that in fact $\{u_n\}, \{v_n\} \in \ell_2(\mathbb{Z}\backslash\{0\})$ are both analytic in a neighbourhood of $p = 0$. The behaviour of $q^{(2)}$ near $p = 0$ is then proved as in the non-resonant case, but we have now to take account of $q^{(1)}$, since the coefficient in $\mathcal{M}_1$ for $n = N$ (see the definition (2.18)) has a pole at $p = 0$. But from (4.1) one has

$$\frac{\mathrm{e}^{-2r\sqrt{n/N-\mathrm{i}p}}}{(2\pi)^3 r^2} q_n^{(2)}(0) = -\frac{2\mathrm{i}\sqrt{2\pi}}{r(1+\sqrt{n/N})}\left[\frac{(2\pi)^{\frac{3}{2}}-1}{(2\pi)^{\frac{3}{2}}}\,\mathrm{e}^{-r\sqrt{n/N}} - \mathrm{e}^{-r}\right]$$

so that $q_N^{(1)}(0) = \mathrm{i}\sqrt{2\pi}$.

□

6. Complete ionization in the generic case

Summing up the results about the behaviour of the Laplace transforms $\tilde{q}^{(j)}(p)$, $j = 1, 2$, we can state the following:

Theorem 6.1. *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (2.11) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q^{(j)}(t)| \leqslant A_j t^{-\frac{3}{2}} + R_j(t) \tag{6.1}$$

where $A_j > 0$ and $R_j(t)$ has an exponential decay, $R_j(t) \sim C_j\,\mathrm{e}^{-B_j t}$ for some $B_j > 0$. Moreover the system shows asymptotic complete ionization and, as $t \to \infty$,

$$|\theta(t)| = |(\varphi_{\alpha(0)}, \Psi_t)| \leqslant D t^{-\frac{3}{2}} + E(t)$$

where $D > 0$ and $E(t)$ has an exponential decay.

Proof. Propositions 3.3, 4.1 and 5.1 guarantee that $\tilde{q}(p)$ is analytic on the closed right half plane, except branch point singularities on the imaginary axis at $p = \mathrm{i}\omega n$, $n \in \mathbb{Z}$. Therefore we can choose an integration path for the inverse of Laplace transform of $\tilde{q}(q)$ along the imaginary axis like in [4] and the result is a straightforward consequence of the behaviour of $q^{(j)}(p)$ around the branch points given by proposition 5.1 (see, e.g., the proof of theorem 3.1 in [3]).

The Laplace transform of $\theta(t)$ can be expressed in the following way (see, e.g., proposition 2.1 in [3]):

$$\tilde{\theta}(p) = \tilde{Z}(p) + \tilde{Z}_1(p)\tilde{q}^{(1)}(p) + \tilde{Z}_2(p)\tilde{q}^{(2)}(p)$$

where $\tilde{Z}(p)$ is analytic on the closed right half plane and $\tilde{Z}_j(p)$ has only a branch point at the origin of the form $a_j + b_j\sqrt{p}$. Hence $\tilde{\theta}(p)$ has the same singularities as $\tilde{q}(p)$ and then its asymptotic behaviour coincides with that of $q(t)$. □

In the following we shall prove a stronger result about complete ionization of the system, namely that every state $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state for the operator $H(t)$, i.e. for any $0 < R < \infty$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{d}\tau\, \|F(|\boldsymbol{x}| \leqslant R)U(\tau, 0)\Psi\|^2 = 0 \tag{6.2}$$

where $F(S)$ is the multiplication operator by the characteristic function of the set $S \subset \mathbb{R}^3$ and $U(t, s)$ the unitary two-parameters family associated with $H(t)$ (see (2.5)).

In order to prove (6.2), we first need to study the evolution of a generic initial datum in a suitable dense subset of $L^2(\mathbb{R}^3)$ and then we shall extend the result to every state using the unitarity of the evolution defined by (2.5) (see, e.g., [9]).

Proposition 6.1. *Let $\Psi \in C_0^\infty(\mathbb{R}^3\backslash\{0, \boldsymbol{r}\})$ a smooth function with compact support away from $0, \boldsymbol{r}$ and $q^{(j)}(t)$ be the solutions of equations (2.7) and (2.8) with initial condition $\Psi_0 = \Psi$. If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (2.11) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q^{(j)}(t)| \leqslant A_j t^{-\frac{3}{2}} + R_j(t) \tag{6.3}$$

where $A_j > 0$ and $R_j(t)$ has an exponential decay, $R_j(t) \sim C_j\,\mathrm{e}^{-B_j t}$ for some $B_j > 0$.

Proof. The estimate on the behaviour for large time contained in section 2 still applies, so that $\tilde{q}^{(j)}(p)$ is analytic $\forall p$ with $\mathrm{Re}(p) > b_0$.

Hence we can consider the Laplace transforms of equations (2.7) and (2.8), which have the form (2.16) and (2.17) with

$$G_1(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-pt} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\hat{\Psi}(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}k^2 t}$$

$$G_2(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-pt} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\hat{\Psi}(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}(k^2-\boldsymbol{k}\cdot\boldsymbol{r})t}$$

where $\hat{\Psi}(\boldsymbol{k})$ is the Fourier transform of Ψ.

Since for every smooth function Ψ with compact support, $\hat{\Psi}(\boldsymbol{k})$ is a smooth function with an exponential decay as $k \to \infty$, so that $G_j(p)$ has the same singularities as in the case already studied, i.e. a branch point at the origin of the form $a(p) + b(p)\sqrt{p}$. □

Theorem 6.2. *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (2.11) with respect to $\mathcal{T}$, every $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state of $H(t)$, i.e. it satisfies (6.2). Moreover the point spectrum of the Floquet operator associated with $H(t)$,*

$$K \equiv -\mathrm{i}\frac{\partial}{\partial t} + H(t)$$

is empty.

Proof. The proof follows from unitarity of the evolution and the explicit expression (2.6), together with proposition 6.1 (see the proof of theorem 3.2 in [3]). The absence of eigenvalues of the Floquet operator is a straightforward consequence: every eigenvector of K is of the form $\mathrm{e}^{\mathrm{i}\beta t}\chi(\boldsymbol{x}, t)$, where $\beta \in \mathbb{R}$ and χ is periodic in time, hence it cannot satisfy (6.2). □

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